Alternating Series

- **Use the Alternating Series Test to determine whether an infinite series converges.**
- **Use the Alternating Series Remainder to approximate the sum of an alternating series.**
- **Classify a convergent series as absolutely or conditionally convergent.**
- **Rearrange an infinite series to obtain a different sum.**

Alternating Series

So far, most series you have dealt with have had positive terms. In this section and the following section, you will study series that contain both positive and negative terms. The simplest such series is an **alternating series,** whose terms alternate in sign. For example, the geometric series

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n}$$
$$= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$$

is an *alternating geometric series* with $r = -\frac{1}{2}$. Alternating series occur in two ways: either the odd terms are negative or the even terms are negative.

THEOREM 9.14 Alternating Series Test

Let $a_n > 0$. The alternating series

$$\sum_{n=1}^{\infty}(-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

converge if the following two conditions are met.

1. $\lim_{n\to\infty} a_n = 0$
2. $a_{n+1} \le a_n$, for all n

Proof Consider the alternating series $\Sigma(-1)^{n+1}a_n$. For this series, the partial sum (where $2n$ is even)

$$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \cdots + (a_{2n-1} - a_{2n})$$

has all nonnegative terms, and therefore $\{S_{2n}\}$ is a nondecreasing sequence. But you can also write

$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n}$$

which implies that $S_{2n} \le a_1$ for every integer n. So, $\{S_{2n}\}$ is a bounded, nondecreasing sequence that converges to some value L. Because $S_{2n-1} - a_{2n} = S_{2n}$ and $a_{2n} \to 0$, you have

$$\lim_{n\to\infty} S_{2n-1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n}$$
$$= L + \lim_{n\to\infty} a_{2n} = L.$$

Because both S_{2n} and S_{2n-1} converge to the same limit L, it follows that $\{S_n\}$ also converges to L. Consequently, the given alternating series converges.

NOTE The second condition in the Alternating Series Test can be modified to require only that $0 < a_{n+1} \le a_n$ for all n greater than some integer N.

EXAMPLE 1 Using the Alternating Series Test

NOTE The series in Example 1 is called the *alternating harmonic series*—more is said about this series in Example 7.

Determine the convergence or divergence of $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$.

Solution Note that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$. So, the first condition of Theorem 9.14 is satisfied. Also note that the second condition of Theorem 9.14 is satisfied because

$$a_{n+1} = \frac{1}{n+1} \le \frac{1}{n} = a_n$$

for all n. So, applying the Alternating Series Test, you can conclude that the series converges.

EXAMPLE 2 Using the Alternating Series Test

Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{n}{(-2)^{n-1}}$.

Solution To apply the Alternating Series Test, note that, for $n \ge 1$,

$$\begin{aligned} \frac{1}{2} &\le \frac{n}{n+1} \\ \frac{2^{n-1}}{2^n} &\le \frac{n}{n+1} \\ (n+1)2^{n-1} &\le n2^n \\ \frac{n+1}{2^n} &\le \frac{n}{2^{n-1}}. \end{aligned}$$

So, $a_{n+1} = (n+1)/2^n \le n/2^{n-1} = a_n$ for all n. Furthermore, by L'Hôpital's Rule,

$$\lim_{x\to\infty} \frac{x}{2^{x-1}} = \lim_{x\to\infty} \frac{1}{2^{x-1}(\ln 2)} = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{n}{2^{n-1}} = 0.$$

Therefore, by the Alternating Series Test, the series converges.

EXAMPLE 3 Cases for Which the Alternating Series Test Fails

NOTE In Example 3(a), remember that whenever a series does not pass the first condition of the Alternating Series Test, you can use the nth-Term Test for Divergence to conclude that the series diverges.

a. The alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n} = \frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \cdots$$

passes the second condition of the Alternating Series Test because $a_{n+1} \le a_n$ for all n. You cannot apply the Alternating Series Test, however, because the series does not pass the first condition. In fact, the series diverges.

b. The alternating series

$$\frac{2}{1} - \frac{1}{1} + \frac{2}{2} - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \cdots$$

passes the first condition because a_n approaches 0 as $n \to \infty$. You cannot apply the Alternating Series Test, however, because the series does not pass the second condition. To conclude that the series diverges, you can argue that S_{2N} equals the Nth partial sum of the divergent harmonic series. This implies that the sequence of partial sums diverges. So, the series diverges.

Alternating Series Remainder

For a convergent alternating series, the partial sum S_N can be a useful approximation for the sum S of the series. The error involved in using $S \approx S_N$ is the remainder $R_N = S - S_N$.

THEOREM 9.15 Alternating Series Remainder

If a convergent alternating series satisfies the condition $a_{n+1} \le a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than (or equal to) the first neglected term. That is,

$$|S - S_N| = |R_N| \le a_{N+1}.$$

Proof The series obtained by deleting the first N terms of the given series satisfies the conditions of the Alternating Series Test and has a sum of R_N.

$$\begin{aligned} R_N = S - S_N &= \sum_{n=1}^{\infty} (-1)^{n+1} a_n - \sum_{n=1}^{N} (-1)^{n+1} a_n \\ &= (-1)^N a_{N+1} + (-1)^{N+1} a_{N+2} + (-1)^{N+2} a_{N+3} + \cdots \\ &= (-1)^N (a_{N+1} - a_{N+2} + a_{N+3} - \cdots) \\ |R_N| &= a_{N+1} - a_{N+2} + a_{N+3} - a_{N+4} + a_{N+5} - \cdots \\ &= a_{N+1} - (a_{N+2} - a_{N+3}) - (a_{N+4} - a_{N+5}) - \cdots \le a_{N+1} \end{aligned}$$

Consequently, $|S - S_N| = |R_N| \le a_{N+1}$, which establishes the theorem.

EXAMPLE 4 Approximating the Sum of an Alternating Series

Approximate the sum of the following series by its first six terms.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{n!}\right) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \cdots$$

Solution The series converges by the Alternating Series Test because

$$\frac{1}{(n+1)!} \le \frac{1}{n!} \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n!} = 0.$$

The sum of the first six terms is

$$S_6 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} = \frac{91}{144} \approx 0.63194$$

and, by the Alternating Series Remainder, you have

$$|S - S_6| = |R_6| \le a_7 = \frac{1}{5040} \approx 0.0002.$$

So, the sum S lies between $0.63194 - 0.0002$ and $0.63194 + 0.0002$, and you have

$$0.63174 \le S \le 0.63214.$$

ECHNOLOGY Later, in ection 9.10, you will be able to show iat the series in Example 4 converges)

$$\frac{e-1}{e} \approx 0.63212.$$

or now, try using a computer to btain an approximation of the sum f the series. How many terms do you eed to obtain an approximation that · within 0.00001 unit of the actual ım?

Absolute and Conditional Convergence

Occasionally, a series may have both positive and negative terms and not be an alternating series. For instance, the series

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$$

has both positive and negative terms, yet it is not an alternating series. One way to obtain some information about the convergence of this series is to investigate the convergence of the series

$$\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|.$$

By direct comparison, you have $|\sin n| \le 1$ for all n, so

$$\left|\frac{\sin n}{n^2}\right| \le \frac{1}{n^2}, \quad n \ge 1.$$

Therefore, by the Direct Comparison Test, the series $\sum \left|\frac{\sin n}{n^2}\right|$ converges. The next theorem tells you that the original series also converges.

> **THEOREM 9.16 Absolute Convergence**
>
> If the series $\Sigma|a_n|$ converges, then the series $\Sigma\, a_n$ also converges.

Proof Because $0 \le a_n + |a_n| \le 2|a_n|$ for all n, the series

$$\sum_{n=1}^{\infty} (a_n + |a_n|)$$

converges by comparison with the convergent series

$$\sum_{n=1}^{\infty} 2|a_n|.$$

Furthermore, because $a_n = (a_n + |a_n|) - |a_n|$, you can write

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$$

where both series on the right converge. So, it follows that $\Sigma\, a_n$ converges. ▬

The converse of Theorem 9.16 is not true. For instance, the **alternating harmonic series**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges by the Alternating Series Test. Yet the harmonic series diverges. This type of convergence is called **conditional.**

> **Definitions of Absolute and Conditional Convergence**
>
> 1. $\Sigma\, a_n$ is **absolutely convergent** if $\Sigma\, |a_n|$ converges.
> 2. $\Sigma\, a_n$ is **conditionally convergent** if $\Sigma\, a_n$ converges but $\Sigma\, |a_n|$ diverges.

EXAMPLE 5 Absolute and Conditional Convergence

Determine whether each of the series is convergent or divergent. Classify any convergent series as absolutely or conditionally convergent.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n} = \frac{0!}{2^0} - \frac{1!}{2^1} + \frac{2!}{2^2} - \frac{3!}{2^3} + \cdots$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = -\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \cdots$

Solution

a. By the nth-Term Test for Divergence, you can conclude that this series diverges.

b. The given series can be shown to be convergent by the Alternating Series Test. Moreover, because the p-series

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right| = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

diverges, the given series is *conditionally* convergent.

EXAMPLE 6 Absolute and Conditional Convergence

Determine whether each of the series is convergent or divergent. Classify any convergent series as absolutely or conditionally convergent.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{3^n} = -\frac{1}{3} - \frac{1}{9} + \frac{1}{27} + \frac{1}{81} - \cdots$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)} = -\frac{1}{\ln 2} + \frac{1}{\ln 3} - \frac{1}{\ln 4} + \frac{1}{\ln 5} - \cdots$

Solution

a. This is *not* an alternating series. However, because

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n(n+1)/2}}{3^n}\right| = \sum_{n=1}^{\infty} \frac{1}{3^n}$$

is a convergent geometric series, you can apply Theorem 9.16 to conclude that the given series is *absolutely* convergent (and therefore convergent).

b. In this case, the Alternating Series Test indicates that the given series converges. However, the series

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\ln(n+1)}\right| = \frac{1}{\ln 2} + \frac{1}{\ln 3} + \frac{1}{\ln 4} + \cdots$$

diverges by direct comparison with the terms of the harmonic series. Therefore, the given series is *conditionally* convergent.

Rearrangement of Series

A finite sum such as $(1 + 3 - 2 + 5 - 4)$ can be rearranged without changing the value of the sum. This is not necessarily true of an infinite series—it depends on whether the series is absolutely convergent (every rearrangement has the same sum) or conditionally convergent.

EXAMPLE 7 Rearrangement of a Series

FOR FURTHER INFORMATION Georg Friedrich Riemann (1826–1866) proved that if $\Sigma\, a_n$ is conditionally convergent and S is any real number, the terms of the series can be rearranged to converge to S. For more on this topic, see the article "Riemann's Rearrangement Theorem" by Stewart Galanor in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com.*

The alternating harmonic series converges to ln 2. That is,

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

(See Exercise 49, Section 9.10.)

Rearrange the series to produce a different sum.

Solution Consider the following rearrangement.

$$\begin{aligned}
&1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \cdots \\
&= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \cdots \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \cdots \\
&= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots\right) = \frac{1}{2}(\ln 2)
\end{aligned}$$

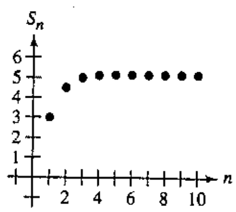
By rearranging the terms, you obtain a sum that is half the original sum.

Exercises for Section 9.5

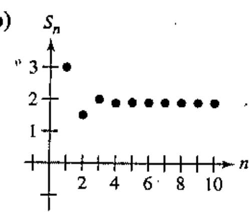
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, match the series with the graph of its sequence of partial sums. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

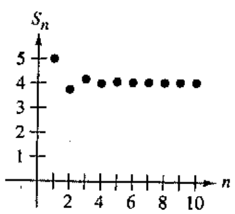
(a)



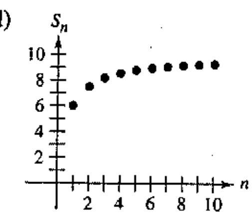
(b)



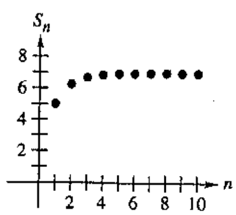
(c)



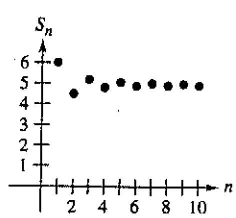
(d)



(e)



(f)



1. $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^2}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}6}{n^2}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n!}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}3}{n!}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{10}{n2^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}10}{n2^n}$

Numerical and Graphical Analysis **In Exercises 7–10, explore the Alternating Series Remainder.**

(a) Use a graphing utility to find the indicated partial sum S_n and complete the table.

n	1	2	3	4	5	6	7	8	9	10
S_n										

(b) Use a graphing utility to graph the first 10 terms of the sequence of partial sums and a horizontal line representing the sum.

(c) What pattern exists between the plot of the successive points in part (b) relative to the horizontal line representing the sum of the series? Do the distances between the successive points and the horizontal line increase or decrease?

(d) Discuss the relationship between the answers in part (c) and the Alternating Series Remainder as given in Theorem 9.15.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} = \frac{1}{e}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} = \sin 1$

In Exercises 11–32, determine the convergence or divergence of the series.

11. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

12. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{2n-1}$

13. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$

14. $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\ln(n+1)}$

15. $\sum_{n=1}^{\infty} \frac{(-1)^{n} n^2}{n^2+1}$

16. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+1}$

17. $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\sqrt{n}}$

18. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+5}$

19. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{\ln(n+1)}$

20. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ln(n+1)}{n+1}$

21. $\sum_{n=1}^{\infty} \sin\frac{(2n-1)\pi}{2}$

22. $\sum_{n=1}^{\infty} \frac{1}{n}\sin\frac{(2n-1)\pi}{2}$

23. $\sum_{n=1}^{\infty} \cos n\pi$

24. $\sum_{n=1}^{\infty} \frac{1}{n}\cos n\pi$

25. $\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!}$

26. $\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!}$

27. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+2}$

28. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{\sqrt[3]{n}}$

29. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n!}{1\cdot 3\cdot 5\cdots(2n-1)}$

30. $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{1\cdot 4\cdot 7\cdots(3n-2)}$

31. $\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n - e^{-n}} = \sum_{n=1}^{\infty}(-1)^{n+1}\operatorname{csch} n$

32. $\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n + e^{-n}} = \sum_{n=1}^{\infty}(-1)^{n+1}\operatorname{sech} n$

In Exercises 33–36, approximate the sum of the series by using the first six terms. (See Example 4.)

33. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3}{n^2}$

34. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}4}{\ln(n+1)}$

35. $\sum_{n=0}^{\infty} \frac{(-1)^{n}2}{n!}$

36. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{2^n}$

In Exercises 37–42, (a) use Theorem 9.15 to determine the number of terms required to approximate the sum of the convergent series with an error of less than 0.001, and (b) use a graphing utility to approximate the sum of the series with an error of less than 0.001.

37. $\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} = \frac{1}{e}$

38. $\sum_{n=0}^{\infty} \frac{(-1)^{n}}{2^n n!} = \frac{1}{\sqrt{e}}$

39. $\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!} = \sin 1$

40. $\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n)!} = \cos 1$

41. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$

42. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n4^n} = \ln\frac{5}{4}$

In Exercises 43–46, use Theorem 9.15 to determine the number of terms required to approximate the sum of the series with an error of less than 0.001.

43. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$

44. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

45. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3-1}$

46. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$

In Exercises 47–62, determine whether the series converges conditionally or absolutely, or diverges.

47. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2}$

48. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$

49. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

50. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

51. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^2}{(n+1)^2}$

52. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n+3)}{n+10}$

53. $\sum_{n=2}^{\infty} \frac{(-1)^{n}}{\ln n}$

54. $\sum_{n=0}^{\infty} (-1)^n e^{-n^2}$

55. $\sum_{n=2}^{\infty} \frac{(-1)^{n}n}{n^3-1}$

56. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1.5}}$

57. $\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!}$

58. $\sum_{n=0}^{\infty} \frac{(-1)^{n}}{\sqrt{n+4}}$

59. $\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1}$

60. $\sum_{n=1}^{\infty} (-1)^{n+1}\arctan n$

61. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$

62. $\sum_{n=1}^{\infty} \frac{\sin[(2n-1)\pi/2]}{n}$

Writing About Concepts

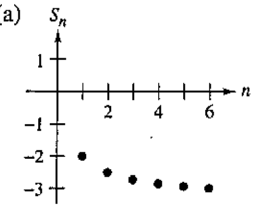
63. Define an alternating series and state the Alternating Series Test.

64. Give the remainder after N terms of a convergent alternating series.

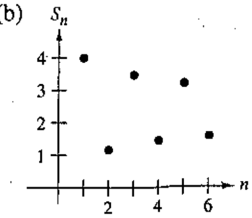
65. In your own words, state the difference between absolute and conditional convergence of an alternating series.

66. The graphs of the sequences of partial sums of two series are shown in the figures. Which graph represents the partial sums of an alternating series? Explain.

(a)



(b)



True or False? **In Exercises 67–70, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

67. If both $\Sigma\, a_n$ and $\Sigma\,(-a_n)$ converge, then $\Sigma\,|a_n|$ converges.

68. If $\Sigma\, a_n$ diverges, then $\Sigma\,|a_n|$ diverges.

69. For the alternating series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, the partial sum S_{100} is an overestimate of the sum of the series.

70. If $\Sigma\, a_n$ and $\Sigma\, b_n$ both converge, then $\Sigma\, a_n b_n$ converges.

In Exercises 71 and 72, find the values of p for which the series converges.

71. $\displaystyle\sum_{n=1}^{\infty} (-1)^n\left(\frac{1}{n^p}\right)$

72. $\displaystyle\sum_{n=1}^{\infty} (-1)^n\left(\frac{1}{n+p}\right)$

73. Prove that if $\Sigma\,|a_n|$ converges, then $\Sigma\, a_n^2$ converges. Is the converse true? If not, give an example that shows it is false.

74. Use the result of Exercise 71 to give an example of an alternating p-series that converges, but whose corresponding p-series diverges.

75. Give an example of a series that demonstrates the statement you proved in Exercise 73.

76. Find all values of x for which the series $\Sigma\,(x^n/n)$ (a) converges absolutely and (b) converges conditionally.

77. Consider the following series.

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{9} + \frac{1}{8} - \frac{1}{27} + \cdots + \frac{1}{2^n} - \frac{1}{3^n} + \cdots$$

(a) Does the series meet the conditions of Theorem 9.14? Explain why or why not.

(b) Does the series converge? If so, what is the sum?

78. Consider the following series.

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n,\; a_n = \begin{cases} \dfrac{1}{\sqrt{n}}, & \text{if } n \text{ is odd} \\[2mm] \dfrac{1}{n^3}, & \text{if } n \text{ is even} \end{cases}$$

(a) Does the series meet the conditions of Theorem 9.14? Explain why or why not.

(b) Does the series converge? If so, what is the sum?

Review **In Exercises 79–88, test for convergence or divergence and identify the test used.**

79. $\displaystyle\sum_{n=1}^{\infty} \frac{10}{n^{3/2}}$

80. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2+5}$

81. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^2}$

82. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

83. $\displaystyle\sum_{n=0}^{\infty} 5\left(\frac{7}{8}\right)^n$

84. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{2n^2+1}$

85. $\displaystyle\sum_{n=1}^{\infty} 100e^{-n/2}$

86. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+4}$

87. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}4}{3n^2-1}$

88. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$

89. The following argument, that $0 = 1$, is *incorrect*. Describe the error.

$$\begin{aligned} 0 &= 0 + 0 + 0 + \cdots \\ &= (1-1) + (1-1) + (1-1) + \cdots \\ &= 1 + (-1+1) + (-1+1) + \cdots \\ &= 1 + 0 + 0 + \cdots \\ &= 1 \end{aligned}$$

90. The following argument, $2 = 1$, is *incorrect*. Describe the error. Multiply each side of the alternating harmonic series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots$$

by 2 to get

$$2S = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \cdots$$

Now collect terms with like denominators (as indicated by the arrows) to get

$$2S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$$

The resulting series is the same one that you started with. So, $2S = S$ and divide each side by S to get $2 = 1$.

FOR FURTHER INFORMATION For more on this exercise, see the article "Riemann's Rearrangement Theorem" by Stewart Galanor in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com.*

Putnam Exam Challenge

91. Assume as known the (true) fact that the alternating harmonic series

(1) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$

is convergent, and denote its sum by s. Rearrange the series (1) as follows:

(2) $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$

Assume as known the (true) fact that the series (2) is also convergent, and denote its sum by S. Denote by s_k, S_k the kth partial sum of the series (1) and (2), respectively. Prove each statement.

(i) $S_{3n} = s_{4n} + \frac{1}{2}s_{2n}$, (ii) $S \neq s$

This problem was composed by the Committee on the Putnam Prize Competition.